Input to State Stabilizing Control Lyapunov Functions for Robust Bipedal Robotic Locomotion

Shishir Kolathaya, Jacob Reher, Ayonga Hereid and Aaron D. Ames

Abstract— **This paper analyzes the input to state stability properties of controllers which stabilize hybrid periodic orbits. Systems that are input to state stable tend to be robust to modeling and sensing uncertainties. The main contribution of this paper is in the construction of control Lyapunov functions that do not just stabilize, but also input to state stabilize a given hybrid system. Bipedal robotic walking, which can be naturally modeled as a hybrid system, is analyzed under this class of controllers. Specifically, we will select a class of controllers via *rapidly exponentially stabilizing control Lyapunov functions* that stabilize bipedal robotic walking; typically modeled as hybrid periodic orbits. We will show with simulation results that given the control Lyapunov functions and the associated set of stabilizing controllers, there exist input to state stabilizing control Lyapunov functions and the associated set of controllers that input to state stabilize the given periodic orbit.**

I. Introduction

Model based controllers are highly sensitive to imperfections in real world implementations. This mismatch is especially relevant in the field of bipedal robotics and can affect controllers adversely through imperfect sensing, inaccurate parameter estimation, input saturation and unmodeled disturbances, to name a few. The notion of *input to state stability* (ISS) [13] captures this uncertainty such that the deviation from the desired output is a function of the deviation from the stabilizing control input. Practical difficulties in the realization of nonlinear feedback controllers on robotic systems place heavy constraints on the ability to increase control gains, and thus to improve convergence rates and tracking errors.

Control Lyapunov functions (CLF), popularized by Artstein and Sontag [12] during the 1980's, enable the use of dynamic programming approaches to obtain optimal control inputs in real-time controllers [4], [3]. The translation of this approach to hybrid systems, especially bipedal robotic systems with underactuation and discrete jumps (impacts), brings with itself a larger challenge. For complex systems such as these, investigating input to state stability (ISS), i.e., studying output perturbations for all kinds of input perturbations seems like an unavoidable task. Indeed, input to state stability of hybrid systems has been studied extensively in literature. Some of the problems addressed are finding a common Lyapunov function [17], [5] and stability under fast switching [10]. There are also interesting properties of several hybrid systems with discrete events that can stabilize continuous unstable dynamics [15].

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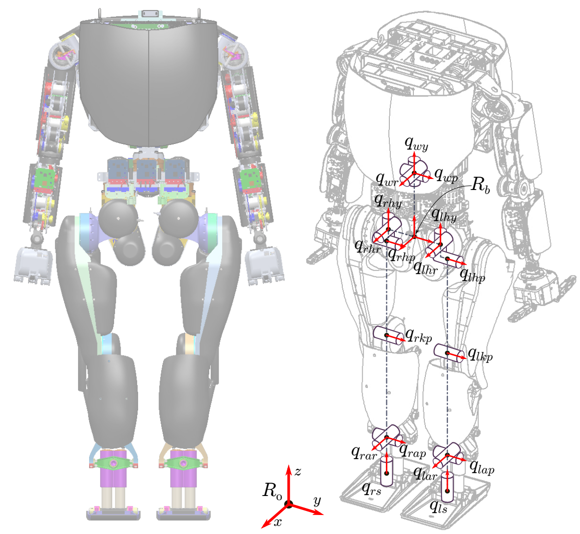

Fig. 1: DURUS robot designed by SRI International in collaboration with AMBER Lab, Pasadena, and Dynamic Robotics Laboratory, Corvallis.

Our quest for desirable stabilization properties under uncertainty lends itself naturally to CLFs. The focal point of this paper is to study and analyze the ISS properties of CLFs for stabilizing nonlinear hybrid systems with affine control inputs; specifically applying to bipedal robots. Through the use of constructions given by Sontag [11], we show that it is indeed possible to find a subset of input to state stabilizing controllers from a given set of stabilizing controllers that are obtained from CLFs. The core advantage is the increase in the number of choices from just one to infinitely many, a necessity for optimal control approaches. We will consider stable walking gaits, i.e., stable hybrid periodic orbits and obtain the ISS properties of these orbits under an ISS based controller. Comparisons are also made between two specific controllers in simulation: feedback linearization and its ISS equivalent, for the humanoid robot DURUS (Fig. 1). Robustness to pushing, uncertain terrain height, and model perturbations are also shown in the analysis.

The paper is structured as follows: Section II contains a brief preliminary on input to state stability. Section III defines *input to state stabilizing control Lyapunov functions*

(ISS-CLFs) and describes the construction process for the set of *input to state stabilizing controllers* via these ISS-CLFs. This section also introduces *rapidly exponentially stabilizing control Lyapunov functions* (RES-CLFs) and the corresponding *rapidly exponential input to state stabilizing control Lyapunov functions* (Re-ISS-CLFs) that are important in the context of hybrid periodic orbits. Section IV will introduce the hybrid robot model of DURUS walking, and the corresponding control methodologies used to realize walking. Section V will introduce the main result and Section VI will show the simulation results and comparisons between a standard stabilizing controller and its ISS equivalent.

II. Preliminaries on Input to State Stability

This section will introduce basic definitions and results related to input to state stability (ISS); for a detailed survey on ISS, see [13]. We consider an affine control system

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

with x taking values in Euclidean space $\mathbb{R}^n$, the input $u \in \mathbb{R}^m$, for some positive integers n, m. The mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are both Lipschitz and $f(0) = 0$. We use a feedback control law

$$u = k(x), \quad k(0) = 0, \tag{2}$$

that makes the closed loop system

$$\dot{x} = f(x) + g(x)k(x), \tag{3}$$

globally asymptotically stable (GAS). We say that a controller, $k(x)$, is stabilizing if it makes the closed loop system (3) GAS. Mathematically, the notion of input/output stability arises from the need to find a feedback control law (2) with the property that the new control system

$$\dot{x} = f(x) + g(x)k(x) + g(x)d, \tag{4}$$

be *input to state stable* where d is called the disturbance input, which belongs to $\mathbb{L}^m_\infty$, i.e., $\|d\|_\infty := \sup_{t\geq 0}\{|d(t)|\}$. Here, $|\,.\,|$ is the Euclidean norm. It is a well known fact that the feedback control law $k(x)$ which achieves statespace stabilization does not necessarily produce input/output stabilization [13]. It is specifically the classes of systems satisfying this property that are of interest to us.

We will define ISS for the dynamics of the form (4). We will be utilizing comparison functions $\mathscr{K}$, $\mathscr{K}\mathscr{L}$ and $\mathscr{K}_\infty$, detailed definitions of which can be found in [13]. It is important to note that the input considered for ISS is the disturbance d. Therefore, all ISS and related definitions are w.r.t. d. Let $x(t, d)$ be the time solution of (4).

Definition 1: *The system* (4) *is input to state stable (ISS-able) if there exists $\beta \in \mathscr{K}\mathscr{L}$, and $\iota \in \mathscr{K}_\infty$ such that*

$$|x(t, x_0, d)| \leq \beta(|x_0|, t) + \iota(\|d\|_\infty), \quad \forall x_0, d, \forall t \geq 0.$$

Definition 2: *The system* (4) *is exponential input to state stable (e-ISSable) if there exists $\beta \in \mathscr{K}\mathscr{L}$, $\iota \in \mathscr{K}_\infty$ and a positive constant $\lambda > 0$ such that*

$$|x(t, x_0, d)| \leq \beta(|x_0|, t)e^{-\lambda t} + \iota(\|d\|_\infty), \quad \forall x_0, d, \forall t \geq 0.$$

Input to state stable Lyapunov functions. A direct consequence of using ISS concepts is the construction of input to state stable Lyapunov functions (ISS-Lyapunov functions).

Definition 3: *A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISS-Lyapunov function for* (4) *if there exist functions $\underline{\alpha}$, $\bar{\alpha}$, α, $\iota \in \mathscr{K}_\infty$ such that $\forall x, d$*

$$\begin{aligned} \underline{\alpha}(|x|) &\leq V(x) \leq \bar{\alpha}(|x|) \\ \dot{V}(x, d) &\leq -\alpha(|x|) + \iota(\|d\|_\infty). \end{aligned} \tag{5}$$

We also have the exponential estimate:

$$\dot{V}(x, d) \leq -cV(x) + \iota(\|d\|_\infty), \tag{6}$$

which is then called the e-ISS-Lyapunov function. It was shown in [13] that a system of the form (4) is ISSable iff it admits a smooth ISS-Lyapunov function. Therefore, for systems that are of interest to us (bipedal robots), we will establish ISS via the construction of ISS-Lyapunov functions.

We say that (1) is smoothly stabilizable, if there is a smooth map $k : \mathbb{R}^n \to \mathbb{R}^m$ with $k(0) = 0$ such that (3) is GAS. (1) is *smoothly input to state stabilizable* (ISSabilizable) if there is a k so that (4) is ISSable. Accordingly, we say that the controller k is an *input to state stabilizing* (ISSabilizing) controller of (1). This can be generalized to define the set of ISSabilizing controllers (i.e., not just one k) via CLFs.

III. Input to state stabilizing control Lyapunov functions

The goal of this section is to derive the set of controllers from CLFs that ISSabilize (1). CLFs are obtained for the control input u, and the ISS conditions are satisfied for the disturbance input d. These CLFs are then called *input to state stabilizing control Lyapunov functions* (ISS-CLFs) [8]. Towards the end of this section, we will derive the set of ISSabilizing controllers from *rapidly exponentially stabilizing control Lyapunov functions* (RES-CLFs) that are important leading into the next section (for hybrid systems).

Input to state stabilizing control Lyapunov functions. We define here a subclass of CLFs that render (1) input to state stable. See [8] for the original definition.

Definition 4: *A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an input to state stabilizing control Lyapunov function (ISS-CLF), if there exists a set of controls $\mathbb{U} \subset \mathbb{R}^m$, and $\underline{\alpha}, \bar{\alpha}, \alpha, \iota \in \mathscr{K}_\infty$ such that $\forall x, d$*

$$\begin{aligned} &\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|) \\ &\inf_{u\in\mathbb{U}}[L_f V(x) + L_g V(x)(u + d)] \leq -\alpha(|x|) + \iota(\|d\|_\infty), \end{aligned}$$

where L_f, L_g are Lie derivatives. Motivated by constructions developed by Sontag, specifically [11, equations (23) and (32)], we can construct ISS-CLFs in the following manner. Consider the following controller which ISSabilizes (1):

$$u = k(x) - \frac{1}{\bar{\varepsilon}} L_g V(x)^T, \tag{7}$$

for some $\bar{\varepsilon} > 0$. Based on this controller, we have the following Lemma which defines a *new* CLF that input to state stabilizes the system (1).

Lemma 1: *The continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined for $\underline{\alpha}, \bar{\alpha}, \alpha \in \mathscr{K}_\infty$ as*

$$\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|) \tag{8}$$
$$\inf_{u\in\mathbb{U}} [L_f V(x) + L_g V(x)u + \alpha(|x|) + \frac{1}{\bar{\varepsilon}} L_g V(x) L_g V(x)^T] \leq 0,$$

is an ISS-CLF $\forall\ \bar{\varepsilon} > 0$.

Proof: After substituting (8) in derivative of V

$$\begin{aligned}\dot{V}(x,u,d) &= L_f V(x) + L_g V(x)u + L_g V(x)d \\ &\leq -\alpha(|x|) - \frac{1}{\bar{\varepsilon}} L_g V(x) L_g V(x)^T + L_g V(x)d.\end{aligned} \tag{9}$$

Since $L_g V(x) \in \mathbb{R}^{1\times m}$, $L_g V L_g V^T = |L_g V|^2 \geq 0$. We have the following inequality after adding and subtracting $\bar{\varepsilon}\frac{\|d\|_\infty^2}{4}$

$$\begin{aligned}\dot{V}(x,d) &\leq -\alpha(|x|) - \left(\frac{1}{\sqrt{\bar{\varepsilon}}}|L_g V(x)| - \sqrt{\bar{\varepsilon}}\frac{\|d\|_\infty}{2}\right)^2 + \bar{\varepsilon}\frac{\|d\|_\infty^2}{4} \\ &\leq -\alpha(|x|) + \bar{\varepsilon}\frac{\|d\|_\infty^2}{4},\end{aligned} \tag{10}$$

which is of the form (5). It can be observed that an excellent way to reduce the effect of $\|d\|_\infty$ is by decreasing $\bar{\varepsilon}$. ■

Rapidly exponentially stabilizing control Lyapunov functions. With the goal of obtaining stronger convergence rates (especially used for hybrid systems like bipedal robots), a *rapidly exponentially stabilizing control Lyapunov function* (RES-CLF) is constructed that stabilizes the output dynamics at a rapidly exponential rate (see [3] for more details) through a user defined $\varepsilon > 0$.

Definition 5: The family of continuously differentiable functions $V_\varepsilon : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *rapidly exponentially stabilizing control Lyapunov function* (RES-CLF) if there exists a set of controls $\mathbb{U} \subset \mathbb{R}^m$, and positive constants $c_1, c_2, c_3 > 0$ such that for all $0 < \varepsilon < 1$, x,

$$c_1\|x\|^2 \leq V_\varepsilon(x) \leq \frac{c_2}{\varepsilon^2}\|x\|^2 \tag{11}$$
$$\inf_{u\in\mathbb{U}} [L_f V_\varepsilon(x) + L_g V_\varepsilon(x)u + \frac{c_3}{\varepsilon} V_\varepsilon(x)] \leq 0.$$

Therefore, we can define a class of controllers $\mathbf{K}_\varepsilon$:

$$\mathbf{K}_\varepsilon(x) = \{u \in \mathbb{U} : L_f V_\varepsilon(x) + L_g V_\varepsilon(x)u + \frac{\gamma}{\varepsilon} V_\varepsilon(x) \leq 0\}, \tag{12}$$

which yields the set of control values that satisfies the desired convergence rate.

If a RES-CLF also satisfies the conditions for ISS, then we have *rapidly exponential input to state stabilizing control Lyapunov functions* (Re-ISS-CLF). We therefore have the following Lemma which provides Re-ISS-CLFs.

Lemma 2: *The continuously differentiable function $V_\varepsilon : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined for $c_1, c_2, c_3 > 0$ as*

$$c_1\|x\|^2 \leq V_\varepsilon(x) \leq \frac{c_2}{\varepsilon^2}\|x\|^2 \tag{13}$$
$$\inf_{u\in\mathbb{U}} [L_f V_\varepsilon(x) + L_g V_\varepsilon(x)u + \frac{c_3}{\varepsilon} V_\varepsilon(x) + \frac{1}{\bar{\varepsilon}} L_g V_\varepsilon(x) L_g V_\varepsilon(x)^T] \leq 0,$$

is an Re-ISS-CLF $\forall\ 0 < \varepsilon < 1$, $\bar{\varepsilon} > 0$.

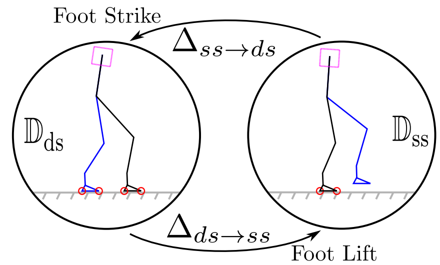

Fig. 2: Hybrid system model for the walking robot DURUS.

Proof of this is similar to (10). We therefore have the set

$$\begin{aligned}\mathbf{K}_{\varepsilon,\bar{\varepsilon}}(x) = \{u \in \mathbb{U} : L_f V_\varepsilon(x) + L_g V_\varepsilon(x)u + \frac{c_3}{\varepsilon} V_\varepsilon(x) \\ + \frac{1}{\bar{\varepsilon}} L_g V_\varepsilon(x) L_g V_\varepsilon(x)^T \leq 0\}.\end{aligned} \tag{14}$$

It can be verified that $\mathbf{K}_{\varepsilon,\bar{\varepsilon}} \subseteq \mathbf{K}_\varepsilon$ (the set obtained from (14) is a subset of (12)). To summarize, for systems of the type (1), we showed here that we can create a set of ISSabilizing controllers via the RES-CLF. The purpose of Re-ISS-CLFs will be more clear in Section IV.

IV. Hybrid Systems

In this section, we will discuss the hybrid control system model of DURUS walking. DURUS is an underactuated 23-DOF bipedal robot designed by SRI International (see Fig. 1) with 15 actuators and 2 springs.

Model. The walking model has two continuous events, double support (ds) and single support (ss), and two discrete events, lift-off (ds $\to$ ss) and foot-strike (ss $\to$ ds), that alternate between each other. We, therefore, have a directed graph, $\Gamma = (\mathbb{V}, \mathbb{E})$, with the set of vertices, $\mathbb{V} = \{\text{ds}, \text{ss}\}$, representing the continuous events and the set of edges, $\mathbb{E} = \{(\text{ds}, \text{ss}), (\text{ss}, \text{ds})\} \subset \mathbb{V} \times \mathbb{V}$, representing the discrete events. A pictorial representation of these individual events and the switch between them are shown in Fig. 2. Given the configuration $q \in \mathbb{Q} \subset \mathbb{R}^n$ of the robot, where $n = 23$, we have the following continuous dynamics for each phase v:

$$\begin{aligned}D(q)\ddot{q} + H(q,\dot{q}) &= B_v u_v + \mathscr{J}_v^T F_v \\ \mathscr{J}_v \ddot{q} + \dot{\mathscr{J}}_v \dot{q} &= 0.\end{aligned} \tag{15}$$

Description of the notations can be found in [6]. Specifically, $u_v \in \mathbb{R}^{m_v}$ is the control input, with $m_{\rm ds} = 9$, $m_{\rm ss} = 15$. Note that we can also represent the dynamics (15) in terms of the state $x := (q, \dot{q})$: $\dot{x} = f_v(x) + g_v(x)u_v$.

Hybrid control system. The hybrid control system model of DURUS is a tuple $\mathscr{HC} = (\Gamma, \mathbb{U}, \mathbb{D}, \mathbb{S}, \Delta, \mathbb{FG})$, with the directed graph $\Gamma = (\mathbb{V}, \mathbb{E})$, the set of inputs $\mathbb{U} = \{\mathbb{U}_{\rm ds}, \mathbb{U}_{\rm ss}\}$, the set of domains $\mathbb{D} = \{\mathbb{D}_{\rm ds}, \mathbb{D}_{\rm ss}\}$, the set of guards $\mathbb{S} = \{\mathbb{S}_{\rm ds}, \mathbb{S}_{\rm ss}\}$, the set of switching functions $\Delta = \{\Delta_{(\rm ds,ss)}, \Delta_{(\rm ss,ds)}\}$, and the set of fields $\mathbb{FG} = \{(f_{\rm ds}, g_{\rm ds}), (f_{\rm ss}, g_{\rm ss})\}$. Note that $\mathbb{U}_v \subset \mathbb{R}^{m_v}$, $\mathbb{D}_v \subset T\mathbb{Q} \times \mathbb{U}_v$, for $v \in \mathbb{V}$. Denote the projection of the domain and guard sets to the states (only) as $\mathbb{S}_v|_x, \mathbb{D}_v|_x$ respectively. More details on the hybrid system model of DURUS, and, in general, of walking robots can be found in [6], [16].

Trajectory tracking control. In the problem of trajectory tracking we drive $k_{1,v}$ relative degree one outputs

$$y_{1,v}(q,\dot{q}) = y^a_{1,v}(q,\dot{q}) - y^d_{1,v}(\alpha_v), \tag{16}$$

and $k_{2,v}$ relative degree two outputs

$$y_{2,v}(q) = y^a_{2,v}(q) - y^d_{2,v}(q,\alpha_v), \tag{17}$$

to zero, with v denoting the domain, α denoting the parameters of the desired trajectory. The total dimension of the outputs $k_{1,v}+k_{2,v}=k_v$ is typically equal to the number of inputs m_v. Since we are interested in seeking a set of control laws, we will use CLF based controllers. Specifically, we will use the CLF based controllers derived from IO linearization.

$$u_{\mathrm{IO}} = M_v^{-1}\left(-\begin{bmatrix} L_{f_v} y_{1,v} \\ L^2_{f_v} y_{2,v}\end{bmatrix} + \mu_v\right),\ M_v = \begin{bmatrix} L_{g_v} y_{1,v} \\ L_{g_v} L_{f_v} y_{2,v}\end{bmatrix}, \tag{18}$$

where $\mu_v \in \mathrm{U}_v \subset \mathbb{R}^{m_v}$ denotes the auxiliary input applied after IO linearization. Let $\eta_v := (y_{1,v}, y_{2,v}, \dot{y}_{2,v})$. Applying (18) for the dynamics of (16), (17) results in the following:

$$\dot{\eta}_v = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_{k_{2,v}} \\ 0 & 0 & 0 \end{bmatrix}}_{F_v} \eta_v + \underbrace{\begin{bmatrix} 1_{k_{1,v}} & 0 \\ 0 & 0 \\ 0 & 1_{k_{2,v}} \end{bmatrix}}_{G_v} \mu_v. \tag{19}$$

$1_{k_{1,v}}, 1_{k_{2,v}}$ are identity matrices of appropriate sizes. The RES-CLF is thus obtained as $V_{\varepsilon,v} := \eta_v^T P_{\varepsilon,v} \eta_v$, where $P_{\varepsilon,v}$ is the solution to the CARE [3, equation (47)].

Re-ISS-CLF. Similar to (4), we are interested in the behavior of DURUS when the controller of the form (18) is applied with a disturbance: $u_{\mathrm{IO}}+d$, resulting in

$$\dot{\eta}_v = F_v\eta_v + G_v\mu_v + G_v M_v d. \tag{20}$$

For the robot model considered (15), the matrix of terms M_v shown above is bounded (also invertible (18)). Hence, by denoting $d_v := M_v d$, we will construct control laws for the auxiliary input μ_v, and then study ISS properties for inputs of the type $\mu_v + d_v$. We have the set of stabilizing controllers obtained from the RES-CLF $V_{\varepsilon,v}$

$$\mathbf{K}_{\varepsilon,v}(\eta_v) = \{\mu_v \in \mathrm{U}_v : \omega_{0,v}(\eta_v) + \omega_{1,v}(\eta_v)\mu_v \le 0\}, \tag{21}$$

and the set of ISSabilizing controllers from $V_{\varepsilon,v}$

$$\begin{aligned}\mathbf{K}_{\varepsilon,\bar{\varepsilon},v}(\eta_v) = \{\mu_v \in \mathrm{U}_v : \omega_{0,v}(\eta_v) + \omega_{1,v}(\eta_v)\mu_v \\ + \frac{1}{\bar{\varepsilon}}\omega_{1,v}(\eta_v)\omega_{1,v}(\eta_v)^T \le 0\},\end{aligned} \tag{22}$$

where

$$\begin{aligned}&\omega_{0,v}(\eta_v) = L_{F_v}V_{\varepsilon,v}(\eta_v) + \frac{\gamma_v}{\varepsilon}V_{\varepsilon,v}(\eta_v), \quad \text{with} \quad 0<\gamma_v<1 \\ &L_{F_v}V_{\varepsilon,v}(\eta_v) = \eta_v^T(F_v^T P_{\varepsilon,v} + P_{\varepsilon,v}F_v)\eta_v \\ &\omega_{1,v}(\eta_v) = L_{G_v}V_{\varepsilon,v}(\eta_v) = 2\eta_v^T P_{\varepsilon,v} G_v.\end{aligned} \tag{23}$$

Note that the set (22) would seem like an overcompensation for the already linearized transverse dynamics (19). Since we are dealing with hybrid systems undergoing nonlinear impacts, we still need the set (22) with the objective of minimizing the disturbance effects via $\bar{\varepsilon}$ (like in (10)). Similar to (10), we have the following derivative of $V_{\varepsilon,v}$:

$$\begin{aligned}\dot{V}_{\varepsilon,v} &= L_{F_v}V_{\varepsilon,v} + L_{G_v}V_{\varepsilon,v}\mu_v + L_{G_v}V_{\varepsilon,v}d_v \\ &\le -\frac{\gamma_v}{\varepsilon}V_{\varepsilon,v} + \frac{\bar{\varepsilon}\|d_v\|^2_\infty}{4}, \quad \text{if} \quad \mu_v(\eta_v) \in \mathbf{K}_{\varepsilon,\bar{\varepsilon},v}(\eta_v).\end{aligned} \tag{24}$$

It can be verified from (24) that $\bar{\varepsilon}$ helps minimize the effects of $\|d_v\|_\infty$ without the need to modify the rate ε.

Partial zero dynamics. As previously mentioned, DURUS is an underactuated robot consisting of two types of dynamics

$$\dot{\eta}_v = F_v\eta_v + G_v\mu_v, \quad \dot{z}_v = \Psi_v(\eta_v, z_v), \tag{25}$$

called the transverse and passive dynamics respectively. In addition, with the convergence of relative degree two outputs $\eta_{2,v} := (y_{2,v}, \dot{y}_{2,v}) \to 0$, we have *partial zero dynamics* [2]

$$\dot{y}_{1,v} = \mu_{1v}, \quad \dot{z}_v = \Psi_v(y_{1,v}, 0, z_v), \tag{26}$$

where μ_{1v} is the first element in μ_v, and the arguments in Ψ_v are separated into three types of coordinates: $(\eta_v, z_v) = (y_{1,v}, \eta_{2,v}, z_v)$. Accordingly, we can define the diffeomorphism $\Phi_v : \mathbb{D}_v|_x \to \mathbb{R}^{2n}$ that maps from x to (η_v, z_v):

$$\Phi_v(x) = \begin{bmatrix} \Phi_{1,v}(x) \\ \Phi_{2,v}(x) \\ \Phi_{3,v}(x) \end{bmatrix} = \begin{bmatrix} y_{1,v}(q,\dot{q}) \\ y_{2,v}(q) \\ \dot{y}_{2,v}(q,\dot{q}) \\ z_v(q,\dot{q}) \end{bmatrix} \tag{27}$$

$$\Phi_v^{\mathrm{PZ}}(x) = \begin{bmatrix} \Phi_{1,v}(x) \\ \Phi_{3,v}(x) \end{bmatrix}, \quad \Phi_v^{\eta_2}(x) = \Phi_{2,v}(x). \tag{28}$$

Partial hybrid zero dynamics. Given the partial zero dynamics (PZD) for each phase $\{\mathrm{ds}, \mathrm{ss}\}$, we can also realize *partial hybrid zero dynamics* (PHZD) if the following hybrid invariance conditions are satisfied

$$\Delta_{(\mathrm{ds,ss})}(\mathbb{PZ}_{\mathrm{ds}} \cap \mathbb{S}_{\mathrm{ds}}|_x) \subset \mathbb{PZ}_{\mathrm{ss}}, \quad \Delta_{(\mathrm{ss,ds})}(\mathbb{PZ}_{\mathrm{ss}} \cap \mathbb{S}_{\mathrm{ss}}|_x) \subset \mathbb{PZ}_{\mathrm{ds}},$$

where $\mathbb{PZ}_v = \{x \in \mathbb{D}_v|_x | \eta_{2,v}(x) = 0\}$, $v \in \mathbb{V}$, is called the *partial zero dynamics surface*. We also define switching functions, Δ_v (not $\Delta_{(\mathrm{ss,ds})}$ or $\Delta_{(\mathrm{ds,ss})}$), for the transformed statespace (from x to (η, z)). For example, $\Delta_{\mathrm{ds}}(\eta_{\mathrm{ds}}, z_{\mathrm{ds}}) := \Phi_{\mathrm{ss}}(\Delta_{(\mathrm{ds,ss})}(\Phi^{-1}_{\mathrm{ds}}(\eta_{\mathrm{ds}}, z_{\mathrm{ds}})))$, which can in turn be split into two components $\Delta^{\eta_2}_{\mathrm{ds}}$, $\Delta^{\mathrm{PZ}}_{\mathrm{ds}}$ corresponding to the coordinates $\eta_{2,v}$ and $(y_{1,v}, z_v)$ respectively. With this new notation, we can reduce the hybrid invariance conditions to the following:

$$\Delta^{\eta_2}_{\mathrm{ds}}(y_{1,\mathrm{ds}}, 0, z_{\mathrm{ds}}) = 0, \quad \Delta^{\eta_2}_{\mathrm{ss}}(y_{1,\mathrm{ss}}, 0, z_{\mathrm{ss}}) = 0. \tag{29}$$

It was shown in [3] that if the PHZD has an exponentially stable periodic orbit, then a Lipschitz continuous feedback control law from the set (21) yields an exponentially stable periodic orbit in the full order dynamics. We will extend this result for inputs of the type (22) along with disturbances[1].

[1] It is important to note that the hybrid invariance conditions shown above need not necessarily be satisfied for the actual system (due to the gap between the assumed and the actual model). If $\hat{\Delta}_v$ is the assumed impact model, the gap can be viewed as: $\Delta_v = \hat{\Delta}_v + d_i$, where d_i is the new impact based disturbance input. This type of characterization was shown in [7], wherein the uncertainty was a function of the model parameters of the bipedal robot. We will ignore this gap here and assume that the disturbances are solely due to the control inputs μ_v.

Periodic orbits and Poincaré maps. Substitution of a control law $k_{\varepsilon,\bar{\varepsilon},v}(\eta_v) \in \mathbf{K}_{\varepsilon,\bar{\varepsilon},v}(\eta_v)$ (from (22)) results in closed loop dynamics of (25). Denote its flow as $\varphi_{t,v}$. For the resulting hybrid dynamics, we have a periodic orbit, if, for some $(\eta^*_{\mathrm{ds}}, z^*_{\mathrm{ds}}) \in \Phi_{\mathrm{ds}}(\mathbb{S}_{\mathrm{ds}}|_x)$, $(\eta^*_{\mathrm{ss}}, z^*_{\mathrm{ss}}) \in \Phi_{\mathrm{ss}}(\mathbb{S}_{\mathrm{ss}}|_x)$, and some $T^*_{\mathrm{ds}}, T^*_{\mathrm{ss}} > 0$,

$$\begin{aligned}(\eta^*_{\mathrm{ss}}, z^*_{\mathrm{ss}}) &= \varphi_{T^*_{\mathrm{ss}},\mathrm{ss}} \circ \Delta_{\mathrm{ds}}(\eta^*_{\mathrm{ds}}, z^*_{\mathrm{ds}}) \\ (\eta^*_{\mathrm{ds}}, z^*_{\mathrm{ds}}) &= \varphi_{T^*_{\mathrm{ds}},\mathrm{ds}} \circ \Delta_{\mathrm{ss}}(\eta^*_{\mathrm{ss}}, z^*_{\mathrm{ss}}).\end{aligned} \tag{30}$$

For $v = \mathrm{ds}$, we have the set of points

$$\mathscr{O}_{\mathrm{ds}} = \{\varphi_{t,\mathrm{ds}}(\Delta_{\mathrm{ss}}(\eta^*_{\mathrm{ss}}, z^*_{\mathrm{ss}})) \in \Phi_{\mathrm{ds}}(\mathbb{D}_{\mathrm{ds}}|_x) | 0 \le t < T^*_{\mathrm{ds}}\}. \tag{31}$$

We can similarly obtain $\mathscr{O}_{\mathrm{ss}}$. Hence, we can define the periodic orbit to be the pair $\mathscr{O} = \{\mathscr{O}_{\mathrm{ds}}, \mathscr{O}_{\mathrm{ss}}\}$, which has the period $T^* = T^*_{\mathrm{ds}} + T^*_{\mathrm{ss}}$. Similar formulations follow for defining a periodic orbit in the PHZD as the pair $\mathscr{O}^{\mathrm{PZ}} = \{\mathscr{O}^{\mathrm{PZ}}_{\mathrm{ds}}, \mathscr{O}^{\mathrm{PZ}}_{\mathrm{ss}}\}$, where the elements are defined via the reduced order flow $\varphi^{\mathrm{PZ}}_{t,v}$. Note that, $T^*_{\mathrm{ds}}, T^*_{\mathrm{ss}}$ (similarly, $T_{\vartheta_{\mathrm{ds}}}, T_{\vartheta_{\mathrm{ss}}}$ for PHZD) are called the times to impact (time to reach the guard) for the corresponding flows in the domain. This can be generalized further to define time to impact functions for states starting from the neighborhood of the orbit. For example, for $(\eta_{\mathrm{ss}}, z_{\mathrm{ss}}) \in \mathbb{B}_* := \mathbb{B}_r(\eta^*_{\mathrm{ss}}, z^*_{\mathrm{ss}}) \cap \Phi_{\mathrm{ss}}(\mathbb{S}_{\mathrm{ss}}|_x)$

$$T_{\mathrm{ds}}(\eta_{\mathrm{ss}}, z_{\mathrm{ss}}) = \min\{t \ge 0 | \varphi_{t,\mathrm{ds}} \circ \Delta_{\mathrm{ss}}(\eta_{\mathrm{ss}}, z_{\mathrm{ss}}) \in \mathbb{B}_*\}. \tag{32}$$

Denote $T := T_{\mathrm{ds}} + T_{\mathrm{ss}}$ (similarly, $T_\vartheta := T_{\vartheta_{\mathrm{ds}}} + T_{\vartheta_{\mathrm{ss}}}$ for PHZD). Given $\varphi_{t,\mathrm{ds}}, \varphi_{t,\mathrm{ss}}$, and $T_{\mathrm{ds}}, T_{\mathrm{ss}}$, we can define the Poincaré map for the initial state $(\eta_{\mathrm{ss}}, z_{\mathrm{ss}}) \in \mathbb{B}_*$ to be

$$\mathbb{P}(\eta_{\mathrm{ss}}, z_{\mathrm{ss}}) = \varphi_{T_{\mathrm{ss}},\mathrm{ss}} \circ \Delta_{\mathrm{ds}} \circ \varphi_{T_{\mathrm{ds}},\mathrm{ds}} \circ \Delta_{\mathrm{ss}}(\eta_{\mathrm{ss}}, z_{\mathrm{ss}}). \tag{33}$$

The Poincaré maps are mapped to and from the guard of the final domain subscript ss. The Poincaré map $\mathbb{P}$ can also be split into two components $\mathbb{P}_{\eta_2}, \mathbb{P}_{\mathrm{PZ}}$ corresponding to the coordinates $\eta_{2,v}$ and $(y_{1,v}, z_v)$ respectively.

Stability of periodic orbits. Stability of periodic orbits can be defined via Poincaré maps [9]. Hence, if the Poincaré map is applied i times on the initial condition $(\eta_{\mathrm{ss}}, z_{\mathrm{ss}})$, then we have the final state as $\mathbb{P}^i(\eta^*_{\mathrm{ss}}, z^*_{\mathrm{ss}})$. We say that the periodic orbit $\mathscr{O}$ is exponentially stable if there is $\xi_p \in (0,1)$, $N_p > 0$ such that for any initial condition $(\eta_{\mathrm{ss}}, z_{\mathrm{ss}}) \in \mathbb{B}_*$, the resulting discrete system satisfies

$$|\mathbb{P}^i(\eta_{\mathrm{ss}}, z_{\mathrm{ss}}) - (\eta^*_{\mathrm{ss}}, z^*_{\mathrm{ss}})| \le N_p \xi_p^i |(\eta_{\mathrm{ss}}, z_{\mathrm{ss}}) - (\eta^*_{\mathrm{ss}}, z^*_{\mathrm{ss}})|.$$

Stability of $\mathscr{O}^{\mathrm{PZ}}$ can also be similarly defined. We will discuss e-ISS of $\mathscr{O}$ next.

V. ISS of Hybrid Periodic Orbits

The goal of this section is to establish e-ISS of $\mathscr{O}$ for inputs of the form: $\mu_v(\eta_v, t) = k_{\varepsilon,\bar{\varepsilon},v}(\eta_v) + d_v(t)$. We will start with the definition of e-ISS for $\mathscr{O}$ (defined via Poincaré maps [14]). Without loss of generality, we will drop the domain subscript notation for the initial states (η, z), guard $\mathbb{S}|_x$, surface $\mathbb{Z}$, and also assume at $(\eta^*_{\mathrm{ss}}, z^*_{\mathrm{ss}}) = (0,0)$. Given that the disturbance d_v is applied in addition to the control law $k_{\varepsilon,\bar{\varepsilon},v}$, the resulting flows $\varphi_{t,v}, \varphi^{\mathrm{PZ}}_{t,v}$, time to impact functions T_v and the Poincaré map $\mathbb{P}$ are now dependent on d_v.

Definition 6: *The periodic orbit $\mathscr{O}$ is e-ISSable (exponential input to state stable) if there is $\xi_p \in (0,1)$, $N_p > 0$ and $\iota_p \in \mathscr{K}_\infty$ such that for any initial condition $(\eta, z) \in \mathbb{B}_*$, the resulting discrete time system satisfies*

$$|\mathbb{P}^i(\eta, z)| \le N_p \xi_p^i |(\eta, z)| + \iota_p(\|d\|_{\mathbb{V}}). \tag{34}$$

e-ISS of $\mathscr{O}^{\mathrm{PZ}}$ is also similarly defined. Note that the disturbance input $\|d\|_{\mathbb{V}}$ is nothing but the maximum of the input disturbances in each domain: $\|d\|_{\mathbb{V}} = \max_{v \in \mathbb{V}} \|d_v\|_\infty$. Given Definition 6, we can now state the main theorem that establishes e-ISS of $\mathscr{O}$.

Theorem 1: *If $\mathscr{O}^{\mathrm{PZ}}$ is e-ISSable, then there exist sufficiently small enough $\varepsilon, \bar{\varepsilon} > 0$ such that for all initial conditions $(\eta, z) \in \mathbb{B}_*$, and for all Lipschitz continuous control laws $k_{\varepsilon,\bar{\varepsilon},v}(\eta_v) \in \mathbf{K}_{\varepsilon,\bar{\varepsilon},v}(\eta_v)$ (22), the full order periodic orbit $\mathscr{O}$ is e-ISSable.*

We will provide a sketch of the proof of Theorem 1. Before proving the theorem, we will first establish some properties of $\mathscr{O}^{\mathrm{PZ}}$. Denote $\zeta := (y_1, z)$. e-ISS of $\mathscr{O}^{\mathrm{PZ}}$ implies that for $d_v = 0$ there exists $r > 0$ such that the restricted Poincaré map $\vartheta : \mathbb{B}_\zeta \to \mathbb{B}_\zeta$, with $\mathbb{B}_\zeta := \mathbb{B}_r(0,0) \cap \Phi^{\mathrm{PZ}}(\mathbb{PZ} \cap \mathbb{S}|_x)$ is exponentially stable i.e., $|\zeta(i)| \le N\xi^i |\zeta(0)|$ for some $N > 0, 0 < \xi < 1$. Therefore, there exists a Lyapunov function V_ϑ, and positive constants b_1, b_2, b_3, b_4 such that

$$\begin{aligned} b_1|\zeta|^2 \le V_\vartheta(\zeta) \le b_2|\zeta|^2 \\ V_\vartheta(\vartheta(\zeta)) - V_\vartheta(\zeta) \le -b_3|\zeta|^2 \\ |V_\vartheta(\zeta) - V_\vartheta(\zeta')| \le b_4|\zeta - \zeta'|.(|\zeta| + |\zeta'|). \end{aligned} \tag{35}$$

We have the following Lemma (required for Theorem 1):

Lemma 3: *Let $\mathscr{O}^{\mathrm{PZ}}$ be e-ISSable. Given the Lipschitz continuous control law $k_{\varepsilon,\bar{\varepsilon},v}(\eta_v) \in \mathbf{K}_{\varepsilon,\bar{\varepsilon},v}(\eta_v)$ (22) that renders the transverse dynamics (20) e-ISSable in the continuous dynamics, then there exist constants $A_1, A_2, D_1, D_2 > 0$, and $\iota \in \mathscr{K}_\infty$ such that for all $(\eta, z) = (y_1, \eta_2, z) \in \mathbb{B}_*$*

$$|T(\eta, z) - T_\vartheta(y_1, z)| \le A_1|\eta_2| + D_1\|d\|_{\mathbb{V}} \tag{36}$$

$$|\mathbb{P}_{\mathrm{PZ}}(\eta, z) - \vartheta(y_1, z)| \le A_2|\eta_2| + D_2\|d\|_{\mathbb{V}}. \tag{37}$$

Proof of Lemma 3 will be omitted due to space constraints, but a similar proof can be found in [7, Lemma 7], wherein the disturbance input was modeled as a function of parameter uncertainty. We will now prove Theorem 1.

Proof: [Proof of Theorem 1] We start by picking a suitable value of ε, as shown by [3, Theorem 2] that yields exponential convergence under a zero disturbance. In order to establish e-ISS of $\mathscr{O}$, it is sufficient to show that the Poincaré map $\mathbb{P}$ is e-ISS [14]. Hence, the goal now is to obtain an ISS-Lyapunov function of the form (6) for the Poincaré map.

For the Re-ISS-CLF V_ε (domain subscript ss is suppressed), denote its reduced Lyapunov function (of only η_2 coordinates) and restriction to the switching surface by V_{ε,η_2}. It can be verified that the matrix P_ε can be separated into two block matrices, with the latter being the matrix used to obtain the Lyapunov function V_{ε,η_2}. We define the following candidate Lyapunov function for some $\sigma > 0$:

$$V_P(\eta, z) = V_\vartheta(\zeta) + \sigma V_{\varepsilon,\eta_2}(\eta_2), \tag{38}$$

defined on B_*. Given the initial state (η, z), We have the value V_{ε,η_2} after the first return on the Poincaré section as

$$V_{\varepsilon,\eta_2}(\mathbb{P}_{\eta_2}(\eta,z)) \leq \frac{c_{2,\mathrm{ss}}}{\varepsilon^2}|\eta_{2,\mathrm{ss}}(T_{\mathrm{ss}})|^2, \tag{39}$$

where $\frac{c_{2,\mathrm{ss}}}{\varepsilon^2} = \lambda_{\max}(P_\varepsilon)$ is the maximum eigenvalue of $P_\varepsilon (= P_{\varepsilon,\mathrm{ss}})$ (domain ss is suppressed). Further substitution yields the following inequality for some constants A_3,A_4,$D'_\eta > 0$:

$$V_{\varepsilon,\eta_2}(\mathbb{P}_{\eta_2}(\eta,z)) \leq A_3|\eta_2|^2 + A_4|\eta_2|\|d\|_{\mathbb{V}} + D'_\eta\|d\|^2_{\mathbb{V}}.$$

See [7, eqn. (87)] for a similar derivation. Therefore, we have

$$\begin{aligned} V_{\varepsilon,\eta_2}&(\mathbb{P}_{\eta_2}(\eta,z)) - V_{\varepsilon,\eta_2}(\eta_2) \\ &\leq A_3|\eta_2|^2 + A_4|\eta_2|\|d\|_{\mathbb{V}} + D'_\eta\|d\|^2_{\mathbb{V}} - c_1|\eta_2|^2, \end{aligned}$$

where $c_1 = \lambda_{\min}(P_\varepsilon)$. By using (37), we have:

$$\begin{aligned} |\mathbb{P}_{\mathrm{PZ}}(\eta,z)| &= |\mathbb{P}_{\mathrm{PZ}}(\eta,z) - \vartheta(\zeta) + \vartheta(\zeta) - \vartheta(0)| \\ &\leq A_2|\eta_2| + D_2\|d\|_{\mathbb{V}} + L_\vartheta|\zeta|, \end{aligned} \tag{40}$$

where L_ϑ is the Lipschitz constant of $\vartheta(\zeta)$. From (35)

$$\begin{aligned} V_\vartheta(\mathbb{P}_{\mathrm{PZ}}(\eta,z)) - V_\vartheta(\vartheta(\zeta)) \leq b_4(A_2|\eta_2| + D_2\|d\|_{\mathbb{V}}) \\ (A_2|\eta_2| + D_2\|d\|_{\mathbb{V}} + (L_\vartheta + N\xi)|\zeta|). \end{aligned} \tag{41}$$

Rest of the proof is similar to [7, equations (91) to (96)], where the final bounds on the states (η, z) are obtained (for a small enough $\bar{\varepsilon}$) that ensure e-ISS of $\mathcal{O}$. ■

Remarks on e-ISS of $\mathcal{O}^{\mathrm{PZ}}$. In Theorem 1, it was assumed that the reduced periodic orbit $\mathcal{O}^{\mathrm{PZ}}$ is e-ISSable. This may seem like a strong assumption, but, for the dynamics of the form (26), the disturbance input affects the outputs $y_{1,v}$ in an additive manner. Therefore, it is straightforward to establish e-ISS of $\mathcal{O}^{\mathrm{PZ}}$, even if we start with the assumption that $\mathcal{O}^{\mathrm{PZ}}$ is exponentially stable under no disturbances.

VI. Results

For verification of the improved stabilizing results presented above, we simulate a bipedal robot (DURUS) under various disturbances and observe improvements of the stability of the gait. The generalized coordinates of the robot are described in Fig. 1 (also see [6]) and the continuous dynamics of the bipedal robot is given by (15). The nominal walking gait considered in this simulation study has two phases: single support, and double support, as shown in Fig. 2. A stable reference walking gait is obtained and verified via an offline optimization algorithm [6]. Therefore, based on [3, Theorem 2], there is a small enough ε (observed to be ≤ 0.2) that makes the hybrid periodic orbit exponentially stable. It is important to note that the torque requirements increase with the decrease in ε.

The main objective of performing a perturbation analysis is to test the stability of the walking gait under uncertainties that are as realistic as possible. Therefore, we set torque limits of 250Nm for each joint and apply a modeling error of 10% to the mass-inertial properties of the robot. Specifically the modeling error was enforced on the mass, center of mass and inertial properties of each link. It was assumed that other properties such as links lengths and spring constants are accurate. The nominal stabilizing controller chosen for simulation is IO linearization (18) with the auxiliary input

$$\mu(\eta_v) = \begin{bmatrix} -\frac{1}{\varepsilon}y_{1,v} \\ -\frac{2}{\varepsilon}\dot{y}_{2,v} - \frac{1}{\varepsilon^2}y_{2,v} \end{bmatrix},$$

and the ISSabilizing controller chosen is (as given by (7))

$$u_{\mathrm{ISS}} = u_{\mathrm{IO}} - \frac{1}{\bar{\varepsilon}}L_g V_\varepsilon^T,$$

for the Lyapunov function obtained via IO linearization (18).

Two test cases were considered: lateral push force to the hip for a duration of 0.1s at the beginning of the single support domain, and stepping onto an unknown ground height. Table I shows the comparison for the push force recovery between u_{IO} and u_{ISS} for different values of $\varepsilon, \bar{\varepsilon}$. It can be observed that with u_{ISS} the robot can handle greater push forces. With lower ε, the stability of the robot is affected (due to 10% model error and torque saturations) resulting in poorer performance for $\varepsilon = 0.05$. On the other hand, Fig. 3 shows that the convergence improves as $\bar{\varepsilon}$ is lowered. Fig. 4 shows the Lyapunov function comparisons for the push recovery. Fig. 5 and Fig. 6 show the comparisons for unknown step over different heights. Fig. 7 shows tiles of push recovery (top) and stepping over (bottom) for an ISSabilizing controller. A video link demonstrating the simulations performed on the robot is given in [1].

Controller	IO Gain (ε)	Maximum Allowable Push (N)
IO	0.2	380
	0.1	420
	0.05	395
ISS ($\bar{\varepsilon} = 0.1$)	0.2	380
	0.1	435
	0.05	410
ISS ($\bar{\varepsilon} = 0.01$)	0.2	435
	0.1	435
	0.05	405

TABLE I: Comparison of maximum recoverable push forces in lateral direction [1]. The ISS based controller can handle greater pushes. Also reducing ε leads to instability due to the constraints on model uncertainty and torque limits.

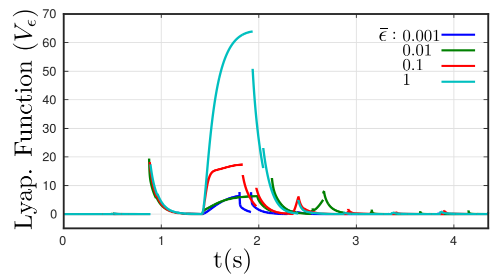

Fig. 3: Comparisons of the Lyapunov function for various values of $\bar{\varepsilon}$ for push recovery. The push force was 350N. The convergence is quicker for decreasing $\bar{\varepsilon}$. The jumps are due to discrete events (impacts).

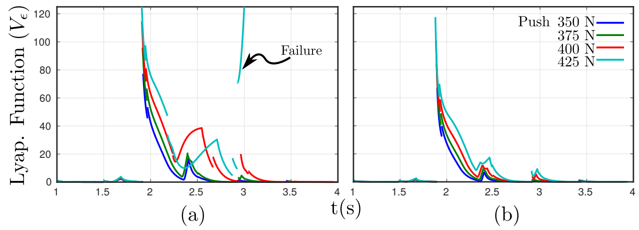

Fig. 4: Push recovery comparison via the Lyapunov functions for IO (a) and ISS (b) based controllers. $\varepsilon = \bar{\varepsilon} = 0.1$. The deviations are lower for the ISS-CLF based controller.

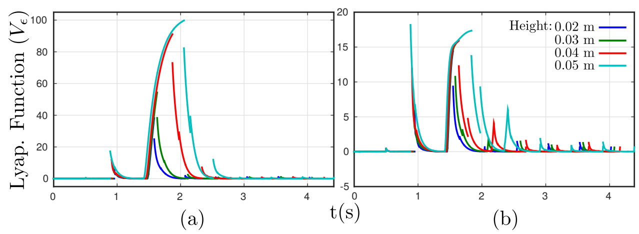

Fig. 5: Step over comparison via Lyapunov functions for IO (a) and ISS (b) based controllers. $\varepsilon = \bar{\varepsilon} = 0.1$. The jumps in the Lyapunov function is lower for ISS-CLF based controller.

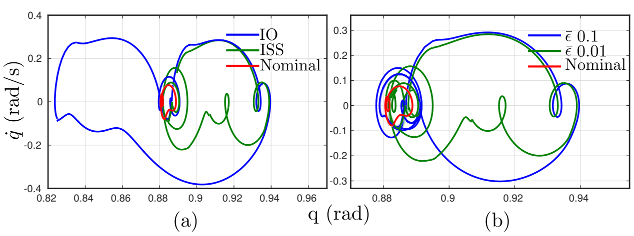

Fig. 6: Walking over 5cm step height. Phase portraits for vertical z position of the torso base are shown here. (a) shows comparison between IO (blue) and ISS (green) and (b) shows the responses for different values of $\bar{\varepsilon}$.

VII. Conclusions

In this work, it was shown how to obtain a class of input to state stabilizing controllers for hybrid systems, given the set of stabilizing controllers. It was shown in the specific case of the bipedal robot DURUS. We obtained the class of input to state stabilizing controllers (22) that adds robustness to the given hybrid periodic orbit $\mathcal{O}$. The simulation results demonstrated that the auxiliary gain $\bar{\varepsilon}$ can be used to restrict the ultimate bound of the outputs without compromising on the convergence rate $\frac{\gamma}{\varepsilon}$ provided by the RES-CLF (21). The methodology shown can be used to realize robust quadratic programs in real time with the end result being input to state stable walking on DURUS.

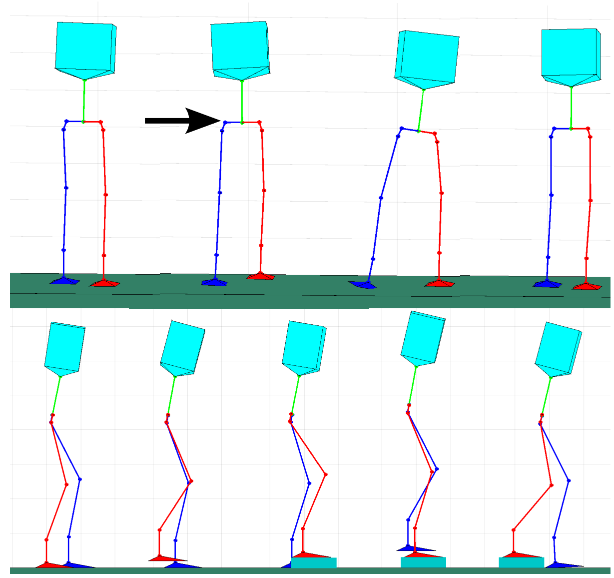

Fig. 7: The top tiles show push (350N) recovery and the bottom tiles show stepping onto an unknown disturbance for an ISSabilizing controller. The IO controller failed for $\varepsilon = 0.2$ and a height of 5cm (shown in video [1]).

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